III

Final Examination

Answer all questions. Each question carries ten points. You should justify your answer and show all details.

1. Let D be the region bounded by the curves $y = 2x^2, y = 6x^2, x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the first quadrant. Evaluate the double integral

$$\iint_D \frac{y(x^2 + 2y^2)}{x^5} \, dA(x, y) \; .$$

2. Consider the triple integral

$$\iiint_\Omega f(x, y, z) \, dV(x, y, z) \; ,$$

where Ω is the region bounded by $x^2 + y^2 + z^2 = 4, \; x + y + z = 2, \; x, y, z \geq 0$. Express it as (a) an integral in $dydxdz$ and (b) an integral in polar coordinates $d\rho d\varphi d\theta$.

3. Let D be the region bounded by the curves $y = x^2$ and $x + y = 12$ and C the boundary of D oriented in the anticlockwise way. Determine the circulation of the field $\mathbf{G} = (3x^2 + y\cos xy)\mathbf{i} + (7x + x\cos xy)\mathbf{j}$ around C.

4. Let R be the half disk $(x - 2)^2 + y^2 \leq 1, \; x, y \geq 0$, in the xy-plane. Find the surface area of the solid obtained by rotating R about the z-axis.

5. Evaluate the surface integral

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma \; ,$$

where S is the part of $z = x^2 + y^2$ pinched between $z = 2, 4$ with normal pointing out and $\mathbf{F} = 3z\mathbf{i} + 5x\mathbf{j} - 2y\mathbf{k}$.

6. Determine the work done by the force $\mathbf{E} = e^{yz}\mathbf{i} + (xze^{yz} + z\cos y)\mathbf{j} + (xye^{yz} + \sin y)\mathbf{k}$ on a person who walks from $A(1, 0, 0)$ to $B(-1, 6\pi^2, 100\pi)$ along the path $t \mapsto (\cos t, 6t^2, 100t)$.

7. Let Ω be the set bounded by $z = 0, y = 1, y = 3$ and $z = 4 - x^2$ and S its boundary. Find the outward flux of the vector field

$$\mathbf{H}(x, y, z) = (x^2 + \cos y)\mathbf{i} + (y + \sin xz)\mathbf{j} + (z + e^x)\mathbf{k}$$

across S.

8. Evaluate the improper integral

$$\int_0^\infty e^{-x^2} x^2 \, dx \; .$$

You may use the formula

$$\int_{-\infty}^\infty e^{-x^2} \, dx = \sqrt{\pi}.$$

9. Let

$$\mathbf{F}(x, y) = \frac{x}{x^2 + y^2} \, \mathbf{i} + \frac{y}{x^2 + y^2} \, \mathbf{j} \; ,$$

and E be the ellipse $(x - 1)^2 + 4y^2 = 3$ with normal pointing out. Find the flux of $\mathbf{F}$ across E.

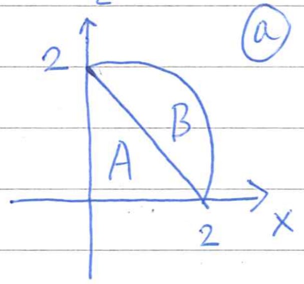
Exam. III.

1. $u = y/x^2 \in [2,6]$, $v = x^2+y^2 \in [1,4]$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -\frac{2y}{x^3} & \frac{1}{x^2} \\ 2x & 2y \end{vmatrix} = \frac{-4y^2}{x^3} - \frac{2}{x} = \frac{-2(2y^2+x^2)}{x^3}.$$

$$\therefore \iint_D \frac{y(x^2+2y^2)}{x^5}\, dA(x,y) = \int_1^4 \int_2^6 \frac{y(x^2+2y^2)}{x^5} \left| \frac{x^3}{-2(2y^2+x^2)} \right| du\, dv$$

$$= \int_1^4 \int_2^6 \frac{1}{2}\frac{y}{x^2}\, du\, dv = \int_1^4 \int_2^6 \frac{u}{2}\, du\, dv = 24. \#$$

2. 

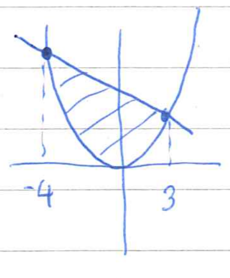
ⓐ over A, Ω is bdd by $y = \sqrt{4-x^2-z^2}$ and $= 2-x-z$
ove B, Ω is bdd by $y=0$ and $\sqrt{4-x^2-z^2}$.

$$\therefore \iiint_\Omega f\, dV = \iint_A \int_{2-x-z}^{\sqrt{4-x^2-z^2}} f\, dy\, dA(x,z)$$

$$+ \iint_B \int_0^{\sqrt{4-x^2-z^2}} f\, dy\, dA(x,z)$$

$$= \int_0^2 \int_0^{2-z} \int_{2-x-z}^{\sqrt{4-x^2-z^2}} f\, dy\, dx\, dz + \int_0^2 \int_{2-z}^{\sqrt{4-z^2}} \int_0^{\sqrt{4-x^2-z^2}} f\, dy\, dx\, dz.$$

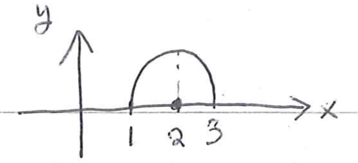
ⓑ $$\iiint_\Omega f\, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_{2/(\sin\varphi\cos\theta + \sin\varphi\sin\theta + \cos\varphi)}^{2} f\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta.$$

3. 

Green's thm

$$\text{circulation} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \int_{-4}^{3} \int_{x^2}^{12-x} 7\, dy\, dx = \cdots \quad \#$$



4. Let $(x, y) = (2+\cos\theta, \sin\theta)$

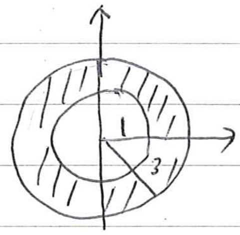
$(x', y') = (-\sin\theta, \cos\theta)$

$|(x', y')| = \sqrt{(-\sin\theta)^2 + (\cos\theta)^2} = 1$

Surface area of torus $= 2\pi \int_0^{2\pi} (2+\cos\theta)\,|(x', y')|\,d\theta = 2\pi \times 2\pi \times 2 = 8\pi^2$

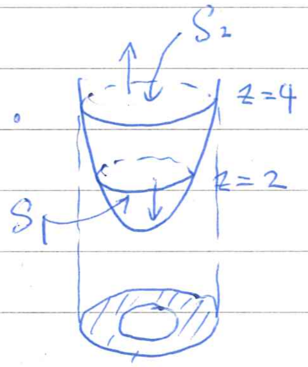
half torus $= 8\pi^2/2 = 4\pi^2$.

need to add the area obtained by rotating the line segment 1 to 3 : $\pi 3^2 - \pi \cdot 1^2 = 8\pi$



$\therefore$ surface area $= 4\pi^2 + 8\pi$ #

5.



$\nabla \times \vec{F} = -2\hat{i} + 3\hat{j} + 5\hat{k}$

$S_2 : x^2+y^2 = 4$ on $z=4$, $\hat{n} = \hat{k}$

$S_1 : x^2+y^2 = 2$ on $z=2$, $\hat{n} = -\hat{k}$

Stokes' thm

$$\left(\iint_S + \iint_{S_1} + \iint_{S_2}\right) \nabla\times\vec{F}\cdot\hat{n} = 0$$

$$\iint_{S_1} \nabla\times\vec{F}\cdot\hat{n} = \iint_{S_1} \nabla\times\vec{F}\cdot(-\hat{k}) = -5 \times \text{area of } S_1 = -5\times\pi(\sqrt{2})^2 = -10\pi$$

$$\iint_{S_2} \nabla\times\vec{F}\cdot\hat{n} = \iint_{S_2} \nabla\times\vec{F}\cdot\hat{k} = 5\times \text{area of } S_2 = 5\times\pi 2^2 = 20\pi$$

$$\therefore \iint_S \nabla\times\vec{F}\cdot\hat{n} = 10\pi - 20\pi = -10\pi.$$

* You may also calculate directly by

$$\iint_S (\nabla\times\vec{F}\cdot\hat{n}) = \int_0^{2\pi}\int_{\sqrt{2}}^{2} (\nabla\times\vec{F}\cdot\hat{n})\, r\,dr\,d\theta \quad \text{etc}$$

but easy to go wrong.

6. the force $\vec{E}$ is conservative and the potential :

$$\Phi(x, y, z) = x e^{yz} + z \sin y .$$

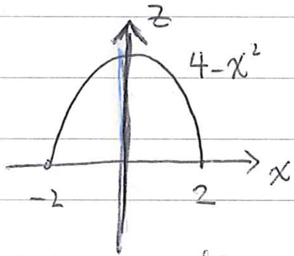
$\therefore$ work done $= \int_A^B \vec{E}\cdot d\vec{r} = \Phi(B) - \Phi(A)$

$$= -e^{6\pi^2 \times 100\pi} + 100\pi \sin 6\pi^2 - 1 .$$

7. Use Divergence thm,

$$\operatorname{div} \vec{H} = \nabla \cdot \vec{H} = 2(x+1)$$

$\therefore$ outward flux $= \iiint_\Omega \operatorname{div} \vec{H}\, dV = 2 \iiint_\Omega (x+1)\, dV .$



a cross section of Ω

$$= \int_1^3 \int_{-2}^{2} \int_0^{4-x^2} 2(x+1)\, dz\, dx\, dy = \cdots \ \#$$

8. $\int_0^a e^{-x^2} x^2\, dx = \int_0^a \left(-\frac{1}{2} e^{-x^2}\right)' x\, dx$

$$= -\frac{1}{2} e^{-x^2} x \Big|_0^a + \frac{1}{2} \int_0^a e^{-x^2}\, dx$$

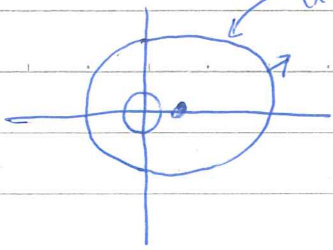
$$\to 0 + \frac{1}{2} \int_0^\infty e^{-x^2}\, dx \quad \text{as } a \to \infty$$

$$\therefore \int_0^\infty e^{-x^2} x^2\, dx = \frac{1}{2} \int_0^\infty e^{-x^2}\, dx$$

$$= \frac{1}{2} \times \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4} .$$

9.

$(x-1)^2+8y^2=20$.



Can't apply Green's thm to the region bdd by E. It is because $\vec{A}$ is not bdd at $(0,0)$.

We let C_r be a little circle around $(0,0)$.
By Green's thm

$(x,y)=(r\cos\theta, r\sin\theta)$
$(x',y')=(-r\sin\theta, r\cos\theta)$
$|(x',y')|=r$
$\hat{n}=(\cos\theta, \sin\theta)$

$$\int_E \vec{A}\cdot\hat{n}\,ds = \int_{C_r} \vec{A}\cdot\hat{n}\,ds$$

$$= \int_0^{2\pi}\left(\frac{r\cos\theta}{r^2}\hat{i}+\frac{r\sin\theta}{r^2}\hat{j}\right)\cdot\hat{n}\; r\,d\theta$$

$$= 2\pi.$$

10. (a) If $F_j=\dfrac{\partial\Phi}{\partial x_j}$, then

$$\frac{\partial F_j}{\partial x_i}=\frac{\partial^2\Phi}{\partial x_i\partial x_j}=\frac{\partial^2\Phi}{\partial x_j\partial x_i}=\frac{\partial F_i}{\partial x_j} \quad \#$$

(b)

$$\frac{\partial}{\partial x_j}\Phi(x)=\int_0^1\frac{\partial}{\partial x_j}\left[F_1(t\vec{x})x_1+F_2(t\vec{x})x_2+\cdots+F_n(t\vec{x})x_n\right]dt$$

$$=\int_0^1\Big[\frac{\partial F_1}{\partial x_j}(t\vec{x})\,t x_1+\frac{\partial F_2}{\partial x_j}(t\vec{x})\,t x_2+\cdots+\frac{\partial F_n}{\partial x_j}(t\vec{x})\,x_n$$

$$+F_j(t\vec{x})\Big]dt$$

$$=\int_0^1\sum_i\left(\frac{\partial F_i}{\partial x_j}(t\vec{x})x_i\right)t+F_j(t\vec{x})\,dt$$

$$=\int_0^1\left(\sum_i\frac{\partial F_j}{\partial x_i}(t\vec{x})x_i\right)t+F_j(t\vec{x})\,dt$$

$$=\int_0^1\frac{d}{dt}\left(F_j(t\vec{x})t\right)dt$$

$$=F_j(t\vec{x})\,t\Big|_{t=0}^{t=1}=F_j(\vec{x}).$$